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LETTER TO THE EDITOR

Lax pair, hidden symmetries, and infinite sequences of conserved currents for self-dual Yang–Mills fields

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Abstract. A Lax pair which linearizes the self-dual Yang–Mills (SDYM) equation is found and shown to be intimately related to the general symmetry problem for SDYM. The linear system is used to derive an invertible recursion operator that produces new infinite sequences of non-local symmetries and associated conservation laws for SDYM.

The integrability properties of the self-dual Yang–Mills (SDYM) equation have been a subject of extensive study over the past fifteen years. As is well known, this nonlinear equation, when properly formulated, displays many of the typical characteristics of an 'integrable' system, such as parametric Bäcklund transformations [1–4], infinite sequences of conservation laws, both non-local [5–7] and local [8], linear system (Lax pair) [9, 10, 6], Painlevé property [11, 12], etc. In particular, the Lax pair was shown to be related both to the presence of a Kac–Moody 'hidden' symmetry [13–15] and to the existence of an infiite number of non-local conserved currents [10].

This letter makes the observation that the SDYM equation can be linearized in more than one way. We propose a new Lax pair for SDYM which allows the relationship between the symmetry and integrability aspects of this equation to become most transparent. This Lax pair is used to construct an invertible recursion operator which produces new infinite sequences of non-local symmetries and associated conservation laws for SDYM. The previously mentioned Kac–Moody symmetry appears naturally as a subsymmetry generated by purely internal transformations.

We write the SDYM in gauge-invariant form [16, 17, 6]:

$$F(J) \equiv D_{\bar{y}}(J^{-1}J_y) + D_{\bar{z}}(J^{-1}J_z) = 0 \qquad (1)$$

(where we use the notation $J_y = D_y J \equiv \partial J/\partial y$, etc, for partial derivatives). The variables y, z, $\bar{y}$, $\bar{z}$ are constructed from the coordinates of an underlying complexified Euclidean space in such a way that $\bar{y}$ and $\bar{z}$ become the complex conjugates of y and z, respectively, when the above space is real. The variable J is, in general, an N-dimensional complex, non-singular matrix. For real SU(N) gauge theory, J is required to be a Hermitian SL(N, C) matrix in real space.

Let $J' = J + \alpha Q(J)$ be an infinitesimal symmetry transformation, i.e. one which leaves equation (1) invariant. Here $Q(J)$ is a functional which may be local or non-local

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in J, while α is an infinitesimal parameter. The symmetry condition in order that $F(J')=0$, whenever $F(J)=0$, is

$$D_{\bar{y}}\{J^{-1}[Q(J)J^{-1}]_y J\}+D_{\bar{z}}\{J^{-1}[Q(J)J^{-1}]_z J\}=0. \tag{2}$$

One often says that the functional $Q(J)$ is a symmetry characteristic for (1).

Equation (2) has been solved for the particularly simple case of point symmetries by using isovector techniques [2, 3]. Moreover, the internal symmetry: $Q(J)=JM$, where M is a constant matrix, serves as a basis for constructing the Kac–Moody 'hidden' symmetry of SDYM [13–15]. We will presently extend the invariance group by adding infinite sequences of symmetries associated with coordinate transformations. To begin with, we propose the following linearization of SDYM.

Proposition 1. Consider the pair of linear equations for ψ:

$$J(J^{-1}\psi)_{\bar{z}}=\lambda(\psi J^{-1})_y J \qquad J(J^{-1}\psi)_{\bar{y}}=-\lambda(\psi J^{-1})_z J \tag{3}$$

where λ is a complex parameter and J is a matrix function. This system is integrable for ψ if J is a solution of (1): $F(J)=0$. Moreover, if $\psi(J;\lambda)$ is a solution of the linear system (3), for some SDYM field J, then ψ is a symmetry characteristic, i.e. satisfies (2).

Proof. The integrability condition $(J^{-1}\psi)_{\bar{z}\bar{y}}=(J^{-1}\psi)_{\bar{y}\bar{z}}$ yields

$$D_{\bar{y}}[J^{-1}(\psi J^{-1})_y J]+D_{\bar{z}}[J^{-1}(\psi J^{-1})_z J]=0. \tag{4}$$

The integrability condition $\psi_{yz}=\psi_{zy}$ yields (after a lengthy calculation, and by using (4)):

$$[J^{-1}\psi, F(J)]=0.$$

For this to be satisfied independently of ψ, one must have $F(J)=0$. A comparison of (4) and (2) then implies that $\psi(J;\lambda)$ is a symmetry characteristic of (1).

Thus, equations (3) constitute a Lax pair for SDYM, the solution ψ of which pair is a symmetry generator. It is natural to seek an explicit construction of ψ for given J and λ. To this end, we try a Laurent expansion in powers of the parameter λ:

$$\psi(J;\lambda)=\sum_{n=-\infty}^{+\infty}\lambda^n Q^{(n)}(J). \tag{5}$$

Substituting this into equations (3), and equating the coefficients of λ^{n+1}, we obtain the pair of equations:

$$J[J^{-1}Q^{(n+1)}]_{\bar{z}}=[Q^{(n)}J^{-1}]_y J \qquad J[J^{-1}Q^{(n+1)}]_{\bar{y}}=-[Q^{(n)}J^{-1}]_z J. \tag{6}$$

The consistency of these relations requires that both $Q^{(n)}$ and $Q^{(n+1)}$ satisfy (2). Technically speaking, equations (6) are a strong Bäcklund transformation for the symmetry condition (2) of SDYM, for a given solution J of (1). Equations (6) may be rewritten in the form of an invertible non-local recursion operator:

$$Q^{(n+1)}=JD_{\bar{z}}^{-1}\{J^{-1}[Q^{(n)}J^{-1}]_y J\} \qquad Q^{(n-1)}=-D_{\bar{z}}^{-1}\{J[J^{-1}Q^{(n)}]_{\bar{y}}J^{-1}\}J. \tag{7}$$

Starting with a known symmetry $Q^{(0)}(J)$ of SDYM (say, a local symmetry), one may construct an infinite sequence of symmetries $Q^{(n)}(J)$ (where $n=\pm1,\pm2,\pm3,\ldots,\pm\infty$) simply by employing the recursion relations (7). At the same time, the solution (5) of the Lax pair is formally represented as an infinite sum of symmetry characteristics of SDYM.

If the original (untransformed) solution J satisfies $\det J = 1$ and $J^{\dagger} = J$ in real space, the conditions in order that a symmetry $Q(J)$ preserve these properties of J, are $\mathrm{tr}(J^{-1}Q) = 0$ and $Q^{\dagger} = Q$ in real space (where the dagger denotes Hermitian conjugation). Let $Q^{(n)}$ be a characteristic with these properties. In general, neither $Q^{(n+1)}$ nor $Q^{(n-1)}$, as given by equations (7), will be Hermitian. To take care of this problem, we use the fact that the symmetry condition (2) is linear in $Q(J)$, hence the difference of two solutions is again a solution (for the same J). Thus we consider the following recursion relation in place of those of equations (7):

$$Q^{(n+1)} = JD_{\bar{z}}^{-1}\{J^{-1}[Q^{(n)}J^{-1}]_{y}J\} + D_{z}^{-1}\{J[J^{-1}Q^{(n)}]_{\bar{y}}J^{-1}\}J. \tag{8}$$

It is readily verified that this operator preserves the required properties of $Q^{(n)}$ for Hermitian SL(N, C) SDYM solutions.

The recursion operator does more than produce new symmetries. Returning to the symmetry condition (2) we observe that it has the form of a continuity equation which is satisfied for all symmetry characteristics $Q^{(n)}(J)$:

$$D_{\bar{y}}\{J^{-1}[Q^{(n)}(J)J^{-1}]_{y}J\} + D_{\bar{z}}\{J^{-1}[Q^{(n)}(J)J^{-1}]_{z}J\} = 0. \tag{9}$$

We thus obtain an infinite sequence of non-local conservation laws for SDYM, corresponding to the infinite sequence of non-local characteristics $Q^{(n)}(J)$. We note that the conserved 'charges' are linearly dependent upon symmetry characteristics. This feature is new, not present in older conservation laws for SDYM [5, 7], and may suggest that these currents are associated with some underlying Noether structure.

We now study the relationship of our Lax pair (3) to the one known previously [6, 9, 10] for SDYM:

$$X_{\bar{z}} = \lambda(X_{y} + J^{-1}J_{y}X) \qquad X_{\bar{y}} = -\lambda(X_{z} + J^{-1}J_{z}X). \tag{10}$$

We have found a simple algebraic relation which allows one to construct solutions ψ of (3) from solutions X of (10) (but not vice versa) for the same J:

Proposition 2. Let $X(J; \lambda)$ be a solution of equations (10), for a given SDYM solution J. Consider the function $\psi(J; \lambda)$ defined by

$$\psi = JXTX^{-1} \tag{11}$$

where

$$T = f(y + \lambda\bar{z}, z - \lambda\bar{y}, \lambda) \tag{12}$$

is an arbitrary function of the indicated variables. Then, ψ is a solution of equations (3).

Proof. We first note that, according to (12), T satisfies the relations $T_{\bar{z}} = \lambda T_{y}$ and $T_{\bar{y}} = -\lambda T_{z}$. Putting $\phi \equiv XTX^{-1}$, and using equations (10), we find that ϕ satisfies the pair of equations

$$\phi_{\bar{z}} = \lambda(\phi_{y} + [J^{-1}J_{y}, \phi]) \qquad \phi_{\bar{y}} = -\lambda(\phi_{z} + [J^{-1}J_{z}, \phi]).$$

By substituting $\phi = J^{-1}\psi$, we recover the linear system (3) for ψ.

Thus, (11) and (12) constitute a weak, non-auto-Bäcklund transformation which produces solutions of the Lax pair (3) from solutions of the Lax pair (10) (this does *not* imply, however, that *all* solutions of (3) may be obtained in this way). This transformation is of practical value when seeking solutions of (3), considering the fact that several solutions of (10) are known (see, for example, [9] and [10] for results related to the multi-instanton solution). Special solutions ψ of the Lax pair (3) are important since, as we have seen, they yield new hidden symmetries and conservation laws for SDYM.

In concluding this letter, we give examples of new symmetries by constructing a few of them explicitly. The conditions $\det J = 1$ and $J^\dagger = J$ will be assumed throughout.

(1) First, we remark that the known symmetries can be recovered by using our symmetry-generating process. Let us start with the internal symmetry $Q^{(0)}(J) = JM + M^\dagger J$, where M is a constant, traceless matrix. Application of the recursion operator (8) yields, after a straightforward calculation

$$Q^{(1)}(J) = J[P, M] + [M^\dagger, \bar{P}]J$$

where P and $\bar{P}$ are potentials for the SDYM equation, defined by $J^{-1}J_y = P_{\bar{z}}$, $J^{-1}J_z = -P_{\bar{y}}$ and $J_{\bar{y}}J^{-1} = \bar{P}_z$, $J_{\bar{z}}J^{-1} = -\bar{P}_y$ (note that, by the conditions imposed on J, the P and $\bar{P}$ are traceless and Hermitian-conjugately related in *real* space).

Repeated application of the recursion operator, and expansion of the matrix M in the basis of $\mathrm{sl}(N, C)$, yield an infinite set of infinitesimal transformations which constitute the familiar Kac–Moody symmetry of SDYM [13–15]. In the literature [13] this symmetry was found by exploiting the infinitesimal transformation $\delta J = -JXMX^{-1}$, where X is a solution of system (10) and M is an infinitesimal constant matrix. (The connection of the aforementioned transformation with (11) is evident.)

(2) Let us start with the translational symmetry [3] $Q^{(0)}(J) = J_y + J_{\bar{y}}$ (note that $\mathrm{tr}(J^{-1}J_y) = 0$, etc). Application of the recursion operator (8) yields

$$Q^{(1)}(J) = J(P_y + P_{\bar{y}}) + (\bar{P}_y + \bar{P}_{\bar{y}})J$$

and so forth. We thus obtain an infinite sequence of new non-local symmetries and conservation laws; the latter are found by direct substitution of the $Q^{(n)}$ into (9).

(3) The dilational symmetry $Q^{(0)} = yJ_y + zJ_z + \bar{y}J_{\bar{y}} + \bar{z}J_{\bar{z}}$ yields

$$Q^{(1)} = J(yP_y + zP_z + \bar{y}P_{\bar{y}} + \bar{z}P_{\bar{z}}) + (y\bar{P}_y + z\bar{P}_z + \bar{y}\bar{P}_{\bar{y}} + \bar{z}\bar{P}_{\bar{z}})J$$

and so forth.

We work similarly for the remaining coordinate symmetries [2, 3]; i.e., the translational symmetry $Q^{(0)} = J_z + J_{\bar{z}}$, and the 'rotational' symmetry $Q^{(0)} = zJ_y - yJ_z + \bar{z}J_{\bar{y}} - \bar{y}J_{\bar{z}}$.

In summary, we have proposed a linearization of SDYM which makes the connection between symmetry and integrability most transparent. The Lax pair was used to construct an invertible recursion operator which, in turn, produced new hidden non-local symmetries and conservation laws. We have discussed possible representations for solutions of the Lax pair, either as infinite sums of symmetry characteristics, or as images, under a weak Bäcklund map, or solutions of the Belavin–Zakharov–Pohlmeyer–Chau linear system. The aforementioned map, being non-surjective, does not yield the general solution of the Lax pair; this probably explains why the older linear system fails to produce the complete symmetry group of SDYM, in contrast to the new one. The solution-generating aspects of the latter system will be explored in future publications.

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